

Algebraic classification of spacetimes using discriminating scalar curvature invariants

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Abstract

The Weyl tensor and the Ricci tensor can be algebraically classified in a Lorentzian spacetime of arbitrary dimensions using alignment theory. Used in tandem with the boost weight decomposition and curvature operators, the algebraic classification of the Weyl tensor and the Ricci tensor in higher dimensions can then be refined utilizing their eigenbivector and eigenvalue structure, respectively. In particular, for a tensor of a particular algebraic type, the associated operator will have a restricted eigenvector structure, and this can then be used to determine necessary conditions for a particular algebraic type. In principle, this analysis can be used to study all of the various algebraic types (and their subclasses) in more detail. We shall present an analysis of the discriminants of the associated characteristic equation for the eigenvalues of an operator to determine the conditions on (the associated) curvature tensor for a given algebraic type. We will describe an algorithm which enables us to completely determine the eigenvalue structure of the curvature operator, up to degeneracies, in terms of a set of discriminants. Since the characteristic equation has coefficients which can be expressed in terms of the scalar polynomial curvature invariants of the curvature tensor, we express these conditions (discriminants) in terms of these polynomial curvature invariants. In particular, we can use the techniques described to study the necessary conditions in arbitrary dimensions for the Weyl and Ricci curvature operators (and hence the higher dimensional Weyl and Ricci tensors) to be of algebraic type **II** or **D**, and create syzygies which are necessary for the special algebraic type to be fulfilled. We are consequently able to determine the necessary conditions in terms of simple scalar polynomial curvature invariants in order for the higher dimensional Weyl and

Ricci tensors to be of type **II** or **D**. We explicitly determine the scalar polynomial curvature invariants for a Weyl or Ricci tensor to be of type **II** or **D** in five dimensions. A number of simple examples are presented to illustrate the calculational method and the power of the approach. In particular, we will present a detailed analysis of the important example of a 5 dimensional rotating black ring.

1 Introduction

Higher dimensional Lorentzian spacetimes are of considerable interest in current theoretical physics. Lorentzian spacetimes for which all polynomial scalar invariants constructed from the Riemann tensor and its covariant derivatives are constant are called *CSI* spacetimes [1]. All curvature invariants of all orders vanish in an D -dimensional Lorentzian *VSI* spacetime [2]. The higher dimensional *VSI* and *CSI* degenerate Kundt spacetimes are examples of spacetimes that are of fundamental importance since they are solutions of supergravity or superstring theory, when supported by appropriate bosonic fields [3]. Higher dimensional black hole solutions are also of current interest [4].

The introduction of *alignment theory* [5] has made it possible to algebraically classify any tensor in a Lorentzian spacetime of arbitrary dimensions by boost weight. In particular, the dimension-independent theory of alignment, using the notions of an aligned null direction and alignment order in Lorentzian geometry, can be applied to the tensor classification problem for the Weyl tensor in higher dimensions [5] (thus generalizing the Petrov classifications in four dimensions (4D or $D = 4$)). Indeed, it is possible to categorize algebraically special tensors in terms of their alignment type, with increasing specialization indicated by a higher order of alignment. In practice, a complete tensor classification in terms of alignment type is possible only for simple symmetry types and for small dimensions [5]. However, partial classification into broader categories is still desirable. We note that alignment type suffices for the classification of 4D Weyl tensors, but the situation for higher-dimensional Weyl tensors is more complicated (and different classifications in 4D are not equivalent in higher dimensions). In the higher dimensional classification, the secondary alignment type is also of significance. Further refinement using bivectors is also useful (see below).

The analysis for higher dimensional Weyl tensors can also be applied directly to the classification of higher dimensional Riemann curvature tensors. In particular the higher-dimensional alignment types give well defined categories for the Riemann tensor (although there are additional constraints coming from the extra non-vanishing components). We can also use alignment to classify the second-order symmetric Ricci tensor (which we refer to as Ricci type). The Ricci tensor can also be classified according to its eigenvalue structure. In addition, using alignment theory, the higher dimensional Bianchi and Ricci identities have been computed [6] and a higher dimensional generalization of *Newman-Penrose formalism* has been presented [5].

Another classification can be obtained by introducing *bivectors* [7]. The algebraic classification of the Weyl tensor using bivectors is equivalent to the algebraic classification of the Weyl tensor by boost weight in 4D (i.e., the Petrov classification [8]). However, these classifications are different in higher dimensions. In particular, the algebraic classification using alignment theory is rather coarse, and it may be useful to develop the algebraic classification of the Weyl tensor using bivectors to obtain a more refined classification.

The bivector formalism in higher dimensional Lorentzian spacetimes was developed in [7]. The Weyl bivector operator was defined in a manner consistent with its boost weight decomposition. The Weyl tensor can then be algebraically classified (based, in part, on the eigenbivector problem), which gives rise to a refinement in dimensions higher than four of the usual alignment (boost weight)

classification, in terms of the irreducible representations of the spins. In particular, the classification in five dimensions was discussed in some detail [7].

1.1 Scalar curvature invariants

A *scalar polynomial curvature invariant of order k* (or, in short, a scalar invariant) is a scalar obtained by contraction from a polynomial in the Riemann tensor and its covariant derivatives up to the order k . The Kretschmann scalar, $R_{abcd}R^{abcd}$, is an example of a zeroth order invariant. Scalar invariants have been extensively used in the study of *VSI* and *CSI* spacetimes [1, 2, 3]. In [9] it was proven that a four-dimensional Lorentzian spacetime metric is either *\mathcal{I} -non-degenerate*, and hence completely locally characterized by its scalar polynomial curvature invariants, or *degenerate Kundt*.

In arbitrary dimensions, demanding that all of the zeroth polynomial Weyl invariants vanish implies that the Weyl type is **III**, **N**, or **O** (similarly for the Ricci type). It would be particularly useful to find necessary conditions in terms of simple scalar invariants in order for the Weyl type (or the Ricci type) to be **II** or **D**. *The main goal of this work is the determination of necessary conditions in higher dimensions for algebraic type, and particularly type II (or D), using scalar curvature invariants.*

1.1.1 Scalar invariants in 4D

In 4D, demanding that the complex zeroth order quadratic and cubic Weyl invariants I and J vanish ($I = J = 0$) implies that the Weyl (Petrov) type is **III**, **N**, or **O** [10]. In addition, the Weyl tensor is of type **II** (or more special; e.g., type **D**) if $27J^2 = I^3$.

It is useful to express the Weyl type **II** conditions in non-NP form. The syzygy $I^3 - 27J^2 = 0$ is complex, whose real and imaginary parts can be expressed using invariants of Weyl not containing duals. The real part is equivalent to:

$$-11W_2^3 + 33W_2W_4 - 18W_6 = 0, \quad (1)$$

and the imaginary part is equivalent to:

$$(W_2^2 - 2W_4)(W_2^2 + W_4)^2 + 18W_3^2(6W_6 - 2W_3^2 - 9W_2W_4 + 3W_2^3) = 0, \quad (2)$$

where

$$\begin{aligned} W_2 &= \frac{1}{8}C_{abcd}C^{abcd}, \\ W_3 &= \frac{1}{16}C_{abcd}C^{cd}_{pq}C^{pqab}, \\ W_4 &= \frac{1}{32}C_{abcd}C^{cd}_{pq}C^{pq}_{rs}C^{rsab}, \\ W_6 &= \frac{1}{128}C_{abcd}C^{cd}_{pq}C^{pq}_{rs}C^{rs}_{tu}C^{tu}_{vw}C^{vwab}. \end{aligned} \quad (3)$$

The Ricci type **II** conditions are :

$$s_3^2(4s_2^3 - 6s_2s_4 + s_3^2) - s_4^2(3s_2^2 - 4s_4) = 0, \quad (4)$$

where S_{ab} is the trace-free Ricci tensor $R_{ab} - \frac{1}{4}Rg_{ab}$ and ¹

$$\begin{aligned} s_2 &= \frac{1}{12}S_a{}^b S_b{}^a, \\ s_3 &= \frac{1}{24}S_a{}^b S_b{}^c S_c{}^a, \\ s_4 &= \frac{1}{48}[S_a{}^b S_b{}^c S_c{}^d S_d{}^a - \frac{1}{4}(S_a{}^b S_b{}^a)^2]. \end{aligned} \tag{5}$$

If a spacetime is Riemann type **II**, then not only do the Weyl type **II** and Ricci type **II** syzygies hold, but there are additional alignment conditions (e.g., $C_{abcd}R^{bd}, C_{abcd}R^{be}R_e{}^d$ are of type **II**).

1.2 Overview

The Weyl tensor and the Ricci tensor can be algebraically classified in a Lorentzian spacetime of arbitrary dimensions by boost weight (using alignment theory). A bivector formalism in higher dimensional Lorentzian spacetimes has been developed to algebraically classify the Weyl bivector operator. Used in tandem with the boost weight decomposition, the algebraic classification of the Weyl tensor and the Ricci tensor (based on their eigenbivector and eigenvalue structure, respectively) can consequently be refined. The purpose of this paper is the determination of necessary conditions for the algebraic type of a higher dimensional Weyl tensor or Ricci tensor, and particularly type **II** (or **D**), using scalar curvature invariants.

For a tensor of a particular algebraic type, the associated operator [11] will have a restricted eigenvector structure. For a given curvature operator in arbitrary dimensions, we can consider the eigenvalues of this operator to obtain necessary conditions. In principle, the analysis can be used to study the various subclasses in more detail. In particular, requiring the algebraic type to be **II** or **D** will impose restrictions on the eigenvalues on the operator.

In this paper we shall present an analysis of the discriminants of the associated characteristic equation to determine the conditions on a tensor for a given algebraic type. Since the characteristic equation has coefficients which can be expressed in terms of the scalar polynomial curvature invariants of the operator, we can consequently give conditions on the eigenvalue structure expressed manifestly in terms of these polynomial curvature invariants. We will describe the algorithm which will enable us to completely determine the eigenvalue structure of the curvature, up to degeneracies, in terms of a set of discriminants nD_i . The resulting syzygies (discriminants) can then be written as special scalar polynomial invariants.

In particular, we use the technique to study the necessary conditions in arbitrary dimensions for the Weyl and Ricci curvature operators (and hence the higher dimensional Weyl and Ricci tensors) to be of algebraic type **II** or **D**. We are consequently able to determine the necessary condition(s) in terms of simple scalar polynomial curvature invariant for the higher dimensional Weyl and Ricci tensors to be of type **II** or **D**. We explicitly display the scalar polynomial curvature invariants for a Weyl or Ricci tensor to be of type **II** (or **D**) in 5D.

¹Note the different index notation on the s_i to that in [10], in order to be self-consistent in this paper.

A number of specific results are obtained, which are summarized at the end of the paper. In addition, a number of simple examples are presented, including Einstein spaces, the 5D Schwarzschild spacetime, and 5D space with complex hyperbolic sections. We will also present a detailed analysis of the important example of a 5D rotating black ring [12] which is generally of type **I_i**, but can also be of type **II** or **D**. This example serves to illustrate the calculational method and the power of the approach. In particular, we shall show that the rotating black ring is of type **II** (or type **D**) on the black hole horizon, by showing that a number of discriminants (the \mathcal{CHP} invariants) vanish there.

We briefly discuss the utility of using these methods to study classification problems that also involve differential scalar polynomial curvature invariants constructed from the Riemann tensor and its covariant derivatives, and present a simple illustration. We also make some brief comments on possible future work. In the Appendices we review the Weyl bivector operator (particularly for type **II** or **D**) and present some important discriminants (or syzygies) that are used in the paper.

2 Discriminant Analysis

We can use an analysis of the discriminants of the associated characteristic equation to determine the conditions on a tensor for a given algebraic type. In particular, we shall seek necessary conditions for a higher dimensional Weyl tensor or Ricci tensor to be type **II** or **D**. In principle, the analysis can be used to study the various subclasses in more detail.

For a tensor of a particular algebraic type, the associated operator (acting on a vector space of dimension n)² will have a restricted eigenvector structure. For a given curvature operator, R , we can consider the eigenvalues of this operator to obtain necessary conditions. In particular, requiring the algebraic type to be **II** or **D** (**II/D**) will impose restrictions of the eigenvalues on the operator (e.g., the eigenvalue type ('Segre type') will have to be of a particular form). Crucial in this discussion is the eigenvalue equation or characteristic equation [11]:

$$\det(R - \lambda 1) = 0. \quad (6)$$

This equation is a polynomial equation in λ and the eigenvalues are the roots of this equation. Since the characteristic equation has coefficients which can be expressed in terms of the invariants of R , we can give conditions of the eigenvalue structure expressed manifestly in terms of the invariants of R . Since the invariants of R are polynomial curvature invariants of spacetime, these conditions will be referred to syzygies. Henceforth, we will describe an algorithm which enables us to completely determine the eigenvalue structure of R using the invariants $\text{Tr}(R^k)$, up to degeneracies which will be explained later.³

2.1 Algorithm

The characteristic equation can be expanded to a polynomial equation:

$$f(\lambda) = \det(\lambda 1 - R) = a_0 \lambda^n + a_1 \lambda^{n-1} + \dots a_i \lambda^{n-i} + \dots + a_n. \quad (7)$$

In our case, the coefficients are expressed in terms of invariants of R , using Newton's identities. However, the algorithm which follows applies to any polynomial equation.

Defining the polynomial invariants⁴

$$R_1 \equiv \text{Tr}(R), \quad R_2 \equiv \text{Tr}(R^2), \quad R_3 \equiv \text{Tr}(R^3), \quad \text{etc}, \quad (8)$$

we can generally write the coefficients a_i as a determinant of an $i \times i$ matrix as

² Notation: For a Lorentzian spacetime of dimension D , non-capitalized Latin indices run over $1, \dots, D-2$, n is the dimension of the vector space (or the order of the associated characteristic equation for the eigenvalues) of the curvature operator (for the Ricci curvature tensor $n = D$ and for the Weyl curvature tensor $n = D(D-1)/2$), and capitalized Latin letters are used to denote bivector indices.

³ Since the coefficients in the characteristic equations are written in terms of invariants of the form $\text{Tr}(R^k)$, we do not need to consider Bianchi identities or dimensional dependent identities to simplify the resulting polynomial expressions obtained.

⁴ We will also use the notation $R1 = R_1$, since this is how it will be presented in MAPLE expressions. We have also omitted any numerical coefficients in the definitions of the R_i for convenience here (see later).

follows:

$$a_i = \frac{(-1)^i}{i!} \det \begin{bmatrix} R_1 & 1 & 0 & \cdots & 0 \\ R_2 & R_1 & 2 & \ddots & \vdots \\ R_3 & R_2 & R_1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & (i-1) \\ R_i & \cdots & R_3 & R_2 & R_1 \end{bmatrix}, \quad (9)$$

where $a_0 \equiv 1$ (and $i = 1, \dots, n$). Explicitly, the first six are given by:

$$\begin{aligned} a_0 &= 1, \\ a_1 &= -R_1, \\ a_2 &= \frac{1}{2}R_1^2 - \frac{1}{2}R_2, \\ a_3 &= -\frac{1}{6}R_1^3 + \frac{1}{2}R_2R_1 - \frac{1}{3}R_3, \\ a_4 &= \frac{1}{24}R_1^4 - \frac{1}{4}R_2R_1^2 + \frac{1}{3}R_3R_1 + \frac{1}{8}R_2^2 - \frac{1}{4}R_4, \\ a_5 &= -\frac{1}{120}R_1^5 + \frac{1}{12}R_2R_1^3 - \frac{1}{6}R_3R_1^2 - \frac{1}{8}R_1R_2^2 + \frac{1}{4}R_4R_1 \\ &\quad + \frac{1}{6}R_2R_3 - \frac{1}{5}R_5. \end{aligned} \quad (10)$$

Also note that the order of a_i is i ; i.e., $\mathcal{O}(a_i) \sim R^i$.⁵

Note that if the invariant of highest order, a_n , is zero, i.e., $a_n = 0$, then the eigenvalue equation trivially factorises and we have a zero eigenvalue. Therefore, it is convenient to first check the existence of zero-eigenvalues. In particular, if

$$a_n = a_{n-1} = \dots = a_{n-k} = 0,$$

then there exists a zero eigenvalue of multiplicity $k+1$. If this the case then the polynomial factorises trivially and the order can be reduced. The following procedure can then be simplified accordingly.

In general the polynomial, eq.(7), can be analysed and criteria for the various 'Segre types' can be given. The resulting syzygies are special polynomial invariants which can be used to characterise the various eigenvalue cases; i.e., they are *discriminants*. A complete set of discriminants can be algorithmically found and in the following we will give the algorithm which is found in [13] (see also [14]). The resulting discriminants will be denoted nD_i , nE_i , nF_i etc., where n denotes order of the polynomial, and i is a running index. These discriminants can be given in terms of the coefficients a_i ; however, using Newton's identities we can express them explicitly in terms of the polynomial invariants R_1 , R_2 , etc.

Given the polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + \dots a_ix^{n-i} + \dots + a_n,$$

⁵It is often convenient to analyse the algebraic structure of the trace-free part of the curvature operator \mathbf{R} , \mathbf{S} , where $S_1 = 0$ and the expressions above simplify; e.g., $a_2 = -\frac{1}{2}S_2$, $a_3 = -\frac{1}{3}S_3$, $a_4 = -\frac{1}{4}S_4 + \frac{1}{8}S_2^2, \dots$

we define the $(2n + 1) \times (2n + 1)$ discrimination matrix $Disc(f)$:

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_n & 0 & \cdots & 0 & 0 \\ 0 & na_0 & (n-1)a_1 & \cdots & a_{n-1} & 0 & \cdots & 0 & 0 \\ 0 & a_0 & a_1 & \cdots & a_{n-1} & a_n & & 0 & 0 \\ 0 & 0 & na_0 & \cdots & 2a_{n-2} & a_{n-1} & & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & na_0 & (n-1)a_1 & \cdots & a_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_{n-1} & a_n \end{bmatrix} \quad (11)$$

Consider now the principal minor series, $\{d_1, d_2, d_3, \dots, d_{2n+1}\}$ defined as the determinants:

$$d_k = \det \begin{bmatrix} \text{the submatrix of } Disc(f) \\ \text{formed by the first} \\ k \text{ rows and } k \text{ columns} \end{bmatrix} \quad (12)$$

Let ${}^n D_i = d_{2i}$, $i = 1, \dots, n$, then the *discriminant sequence* of the polynomial $f(x)$ is given by

$$\{{}^n D_1, {}^n D_2, {}^n D_3, \dots, {}^n D_n\}. \quad (13)$$

By expressing the ${}^n D_i$ in terms of the curvature invariants, R_1, R_2 , etc, we can obtain the *primary syzygies* ${}^n D_i$ for the operator R . Note that the order of ${}^n D_i$ is $\mathcal{O}({}^n D_i) = R^{i(i-1)}$.⁶

Sign List. We call $[\text{sign}({}^n D_1), \text{sign}({}^n D_2), \dots, \text{sign}({}^n D_n)]$, where

$$\text{sign}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases}$$

the *sign list* of the sequence $\{{}^n D_1, {}^n D_2, {}^n D_3, \dots, {}^n D_n\}$.

Revised Sign list. Given a sign list $[s_1, s_2, \dots, s_n]$. If this contains any “internal zeros”, i.e., if there is a subsequence $[s_i, 0, 0, \dots, 0, s_j]$, where $s_i \neq 0$ and $s_j \neq 0$, then we replace this subsequence with:

$$[s_i, -s_i, -s_i, s_i, s_i, -s_i, -s_i, s_i, s_i \dots, s_j].$$

The revised sign list will therefore contain no “internal” zeros, but may have zeros at the end. The revised sign list will give us the number of distinct real and complex roots.

⁶To distinguish between the discriminants of different curvature operators (for example, the Weyl operator and the Ricci operator) in a particular application, we shall (where necessary) add an additional index; e.g., ${}^{10}_W D_k$, $k = 1, \dots, 10$, denote the discriminants of the 5D Weyl curvature operator.

Number of real and complex roots. Consider the revised sign list of $\{{}^n D_1, {}^n D_2, {}^n D_3, \dots, {}^n D_n\}$. Let:

$$K = (\text{number of sign changes}), \quad L = (\text{number of non-zero members}),$$

of the revised sign list. Then for $f(x)$:

- the number of distinct *pairs of complex conjugate roots* is K ; and
- the number of distinct *real roots* is $L - 2K$.

If we are not interested in the multiplicities of the eigenvalues, the discriminant sequence, $\{{}^n D_1, {}^n D_2, {}^n D_3, \dots, {}^n D_n\}$ is sufficient. In some cases, this is enough to determine the eigenvalue structure of R , but not always.

Example: trace-free Ricci tensor in 3D. Let R be the 3-dimensional tracefree Ricci tensor; i.e., ⁷

$$R = (S^\alpha_\beta), \quad S^\alpha_\beta = R^\alpha_\beta - \frac{1}{3}R\delta^\alpha_\beta. \quad (14)$$

This implies,

$$R_1 = 0, \quad R_2 = S^\alpha_\beta S^\beta_\alpha, \quad R_3 = S^\alpha_\beta S^\beta_\delta S^\delta_\alpha.$$

Consequently,

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = -\frac{1}{2}S^\alpha_\beta S^\beta_\alpha, \quad a_3 = -\frac{1}{3}S^\alpha_\beta S^\beta_\delta S^\delta_\alpha.$$

Using the procedure above, we get the discriminants:

$${}^3D_1 = 3, \quad (15)$$

$${}^3D_2 = -6a_2 = 3S^\alpha_\beta S^\beta_\alpha, \quad (16)$$

$${}^3D_3 = -4a_2^3 - 27a_3^2 = \frac{1}{2}(S^\alpha_\beta S^\beta_\alpha)^3 - 3(S^\alpha_\beta S^\beta_\delta S^\delta_\alpha)^2. \quad (17)$$

We clearly have ${}^3D_1 > 0$. The possible signs of the discriminants 3D_2 and 3D_3 can now be used to determine the number of real/complex eigenvalues of S^α_β .

1. ${}^3D_3 > 0, {}^3D_2 > 0$: 3 distinct real eigenvalues.
2. ${}^3D_3 > 0, {}^3D_2 \leq 0$: 2 pairs of complex eigenvalues, which is impossible.
3. ${}^3D_3 < 0$: 1 real and 2 complex eigenvalues.
4. ${}^3D_3 = 0, {}^3D_2 > 0$: 2 real eigenvalues (one of them must be of multiplicity 2).
5. ${}^3D_3 = 0, {}^3D_2 < 0$: 2 complex eigenvalues (impossible, since the last eigenvalue must be real and hence, distinct).
6. ${}^3D_3 = 0, {}^3D_2 = 0$: 1 real eigenvalue (which must be equal to zero since S^α_β is tracefree).

Note that in terms of the Segre type, a (tracefree) Ricci type **II/D** is of type $\{21\}$, $\{(1,1)1\}$, $\{3\}$, or simpler. Consequently, if the (tracefree) Ricci is of type **II/D**, or simpler, then ${}^3D_3 = 0$.

⁷ For illustrative purposes we explicitly repeat the definitions and notation here.

Multiple factor sequence. We note that for polynomials of order 4 and more, the discriminants nD_i may not be sufficient to determine the complete eigenvalue structure. For example, for quartics, if we have 2 distinct real roots, then we cannot, using nD_i only, distinguish the cases $(x - \lambda_1)^3(x - \lambda_2)$ and $(x - \lambda_1)^2(x - \lambda_2)^2$. Therefore, we need to go a step further in order to distinguish these cases.

Consider the discriminant matrix $Disc(f)$. Define the submatrices:

$$M(k, l) \equiv \begin{bmatrix} \text{the submatrix of } Disc(f) \text{ formed by} \\ \text{the first } 2k \text{ rows and} \\ \text{first } (2k - 1) \text{ columns} + (2k + l)\text{th column} \end{bmatrix} \quad (18)$$

Then, construct the polynomials:

$$\Delta_k(f) = \sum_{i=0}^k \det[M(n - k, i)] x^{k-i}, \quad (19)$$

for $k = 0, 1, \dots, n - 1$. The sequence $\{\Delta_0(f), \Delta_1(f), \dots, \Delta_{n-1}(f)\}$ is called the multiple factor sequence of $f(x)$ due to the following result [13]:

Lemma 2.1. *If the number of zeros in the revised list of the discriminant sequence of $f(x)$ is k , then $\Delta_k(f) = \text{g.c.d.}(f(x), f'(x))$.*

The greatest common divisor (g.c.d.) of $f(x)$ and $f'(x)$ is thus always in the multiple factor sequence. Indeed, the polynomial $\Delta(f) \equiv \text{g.c.d.}(f(x), f'(x))$ is the repeated part of $f(x)$, because if $\Delta(f)$ has k real roots of multiplicities n_1, n_2, \dots, n_k , and f has m distinct real roots, then f has k real roots of multiplicities $n_1 + 1, n_2 + 1, \dots, n_k + 1$, and $m - k$ simple real roots (similar argument for complex roots). Therefore, by considering $\Delta(f)$ we reduce the multiplicities of all the roots by 1.

We can now consider the discriminants of the polynomial $\Delta(f)$ in the same way as we computed the discriminant sequence of f . We will call the discriminant sequence of $\Delta(f)$ for $\{{}^nE_1, {}^nE_2, {}^nE_3, \dots, {}^nE_k\}$. We can now use these to determine the sign list of the E -sequence, etc. We can repeat this procedure and consider $\Delta(\Delta(f)) = \Delta^2(f)$, $\Delta^3(f)$ etc. These have the relation:

$$\begin{aligned} \Delta^{j-1}(f) &\propto (x - \lambda_1)^{n_1+1} (x - \lambda_2)^{n_2+1} \dots (x - \lambda_k)^{n_k+1} (x - \lambda_{k+1}) \dots (x - \lambda_m), \\ \Delta^j(f) &\propto (x - \lambda_1)^{n_1} (x - \lambda_2)^{n_2} \dots (x - \lambda_k)^{n_k}, \end{aligned} \quad (20)$$

This gives us the following algorithm for determining the root structure (or eigenvalue structure) [13]:

Algorithm for Root Classification

1. Find the discriminant sequence of $f(x)$:

$$\{{}^nD_1, {}^nD_2, {}^nD_3, \dots, {}^nD_n\},$$

and the revised sign list. Find the number of distinct roots by counting sign changes and non-zero elements of the revised sign list. If the revised sign list contains no 0's, stop.

2. If the revised sign list contains k zeros, then compute the $\Delta(f) = \Delta_k(f)$ by the definition for the multiple factor sequence. Then repeat step 1 for $\Delta(f)$.
3. Continue considering the multiple factor sequence $\Delta^2(f), \Delta^3(f), \dots$, until for some j , the revised sign list of $\Delta^j(f)$ contains no zeros.
4. We now compute the number of real/complex distinct roots of $\Delta^j(f)$. We can now determine the roots and multiplicities of $\Delta^{j-1}(f)$, which again enables us to determine the roots and multiplicities of $\Delta^{j-2}(f)$ etc. At the end of this process, we have a complete root classification for $f(x)$.

Note that this procedure will provide us with the discriminants (or syzygies) which gives us a complete eigenvalue classification of *any* operator R . As explained, these discriminants can be expressed in terms of polynomials of the invariants, $R_1 = \text{Tr}(R)$, $R_2 = \text{Tr}(R^2)$, $R_3 = \text{Tr}(R^3)$ etc., of R . In principle, we can use this method to study the necessary conditions on any curvature operator of any specific eigenvalue type.

In particular, we can use the technique to study the necessary conditions of the Weyl and Ricci curvature operators for it to be of algebraic type **II/D**. We note that the condition ${}^n D_n = 0$ will signal a double eigenvalue since the number of eigenvalues is maximum $(n - 1)$. If ${}^n D_{n-1} = 0$ also, then we have maximum $(n - 2)$ eigenvalues, etc. We can utilise this to create syzygies which are necessary for the special algebraic type to be fulfilled.

2.1.1 Type G and I

Types **G** and **I** are both of equal generality with respect to their possible (eigenbivector/eigenvalue) roots structure.

2.2 Type II/D

Now, for a tensor to be of type **II** (or **D**) then the eigenvalues of the corresponding operator need to be of a special form. Since the invariants of a type **II** are the same as for type **D**, we will assume type **D**. The type **D** case possesses an important symmetry, namely a boost isotropy (the tensor, not necessarily the complete spacetime). This is what we will utilise in the following. This implies specific structure for (a) particular discriminant(s), which then gives rise to necessary condition(s) in terms of scalar polynomial curvature invariants.

For the Ricci tensor, we note that a type **D** tensor is of Segre type $\{(1, 1)11\dots 1\}$, or simpler. This implies that the *Ricci operator has at least one eigenvalue of (at least) multiplicity 2. Furthermore, all the eigenvalues are real.*

For the Weyl tensor in D dimensions we can use the bivector operator in [7] where the canonical form of a Weyl type **D** tensor is given (see also Appendix). In particular, for type **D**,

$$C = \text{blockdiag}(M, \Psi, M^t)$$

where M is a $(D - 2) \times (D - 2)$ matrix and Ψ is a square matrix (see Appendix for the explicit form of this matrix). Since the eigenvalues of M and M^t are the same, we have that the *Weyl operator has at least $(D - 2)$ eigenvalues of (at least) multiplicity 2.*

These observations connect the algebraic types to the eigenvalue structure and enables us to construct the necessary discriminants.

2.2.1 Type II/D in 4D

Applying the condition ${}^3D_3 = 0$ for the complex three dimensional Weyl tensor we obtain the complex syzygy $I^3 - 27J^2 = 0$. This is equivalent to the (12th order) real syzygies given by eqns. (1)-(2) from the six dimensional system with ${}^6D_6 = 0$ and ${}^6D_5 = 0$. Applying the condition ${}^4D_4 = 0$ for the four dimensional trace-free Ricci tensor we obtain the (12th order) syzygy given by eqn. (4).⁸

We can also apply these conditions to the full Riemann tensor (to be of type **II/D**, which implies both the Weyl and Ricci tensor are of type **II/D** and aligned). Alternatively, we note that these will also give us syzygies for mixed tensors. For example requiring that Riemann tensor is type **II/D**, implies that both Ricci and Weyl is type **II/D**, but also mixed tensors, like:

$$L_{\mu\nu} = C_{\mu\alpha\nu\beta}R^{\alpha\beta}, \quad M_{\mu\nu} = C_{\mu\alpha\nu\beta}R^{\alpha}_{\delta}R^{\delta\beta}, \quad N_{\mu\nu} = C_{\alpha\mu\lambda\pi}C^{\lambda\pi}_{\beta\nu}R^{\alpha\beta}.$$

The type **II/D** condition therefore implies that we have the syzygy ${}^4D_4 = 0$ for all of $\mathbf{L} = (L^{\mu}_{\nu})$, $\mathbf{M} = (M^{\mu}_{\nu})$ and $\mathbf{N} = (N^{\mu}_{\nu})$.

2.2.2 Type II/D in 5D

For the trace-free Ricci tensor, we note that type **D** has to be of Segre type $\{(1,1)111\}$ or simpler. This implies that 2 eigenvalues are equal, while the remaining has to be real. Therefore, we get the necessary (20th order) syzygy for the trace-free Ricci tensor to be of type **II/D**:

$${}^5D_5 = 0, \quad {}^5D_4 \geq 0, \quad {}^5D_3 \geq 0, \quad {}^5D_2 \geq 0.$$

Result: *The necessary condition for the trace-free Ricci tensor, S , to be of algebraic type **II** (or **D**) in 5D is that the discriminant ${}^5_S D_5$ is zero, so that the related scalar polynomial curvature invariant $\mathcal{D} \equiv {}^5_S D_5 = 0$.⁹*

For the Weyl tensor, we consider the bivector operator **C**. Since the bivector space is 10-dimensional, we get a condition involving a syzygy of order 90! In particular, the type **II** operator has 3 eigenvalues of (at least) multiplicity 2. Therefore, we get the syzygies:

$${}^{10}D_{10} = {}^{10}D_9 = {}^{10}D_8 = 0.$$

Since these polynomial invariants are of particular importance, we will denote them by $\mathcal{C} \equiv {}^{10}_W D_{10}$, $\mathcal{H} \equiv {}^{10}_W D_9$, $\mathcal{P} \equiv {}^{10}_W D_8$, the \mathcal{CHP} Weyl invariants.

Result: *The necessary condition for the Weyl tensor to be of type **II** (or **D**) in 5D is that the scalar polynomial curvature invariants $\mathcal{C} = \mathcal{H} = \mathcal{P} = 0$.*

⁸Note that the numerical coefficients in the definitions of the polynomial invariants in eqns. (1) - (2) and (4) are different; for example, $R_i \propto W_i$ (i.e., $R_2 = 12W_2$, etc., in (2)).

⁹ $\mathcal{D} \equiv {}^5_S D_5$ is given explicitly in Appendix B.3.

These are syzygies of order 90, 72 and 56, respectively. In principle, these can be computed,¹⁰ but it's probably more useful to consider specific metrics¹¹.

A necessary condition can also be found from considering combinations of the Weyl tensor; for example, the operator $T^\alpha_\beta = C^{\alpha\mu\nu\rho}C_{\beta\mu\nu\rho}$. This gives again

$${}^5_T D_5 = 0, \quad {}^5_T D_4 \geq 0, \quad {}^5_T D_3 \geq 0, \quad {}^5_T D_2 \geq 0.$$

Note that ${}^5_T D_5 = 0$ is now a 40th order syzygy (in the Weyl tensor; a 20th order syzygy in the square of the Weyl tensor).¹²

2.2.3 Caveat: A fundamental degeneracy.

We have stressed that the conditions determined are necessary conditions. Indeed, these conditions may not be sufficient. The reason for this is that the characteristic equation for different algebraic types may be identical and consequently the scalar invariants are also identical. For example, it's not possible to distinguish the Segre types $\{(1,1)111\}$ (of Ricci type **D**) and $\{1,11(11)\}$ (of Ricci type **I**). This implies also that the \mathcal{CHP} conditions may be fulfilled in spite of the fact that the spacetime is of type **G** or **I**. An explicit example of this is the following result:

Proposition 2.2. *Assume a 5D spacetime has a Weyl tensor with $SO(2)$ isotropy. Then it fulfills the \mathcal{CHP} syzygies; i.e., $\mathcal{C} = \mathcal{H} = \mathcal{P} = 0$.*

This result can be seen from using the bivector operator and imposing the $SO(2)$ -symmetry. One then sees that this forces that there must be 3 pairs of equal eigenvalues, which implies that there are maximum 7 distinct eigenvalues; hence, the result follows.

This *degeneracy* in the classification is a fundamental problem when considering scalar invariants only. Sometimes these cases can be resolved by considering other invariants; however, there is no guarantee that this can be achieved. Using the invariants only, we can only determine the *eigenvalue type* of the operator. For example, if we find the eigenvalue type to be $\{(11)111\}$, then this can correspond to three Ricci types: $\{(1,1)111\}$, $\{1,11(11)\}$, and $\{2111\}$ because all of these Ricci types have the same eigenvalue type. This is a *fundamental degeneracy* in the classification of tensors using scalar invariants only and has been discussed in earlier papers [9]. Therefore, we need to keep this in mind when using the discriminants.¹³ Note that this degeneracy is a *discrete* degeneracy, unlike the notion of \mathcal{I} -degenerate metrics [9] which require a *continuous* deformation.

¹⁰ We present the expression for \mathcal{P} in Appendix B.4.

¹¹ Using MAPLE it was possible to compute some of these analytically but in practise the expressions are not very useful. However, for specific metrics these are still computable and may give useful results.

¹² This syzygy is presumably not independent of the syzygies involving $\mathcal{C}, \mathcal{H}, \mathcal{P}$, which perhaps suggests that an appropriate algebraic combination of \mathcal{C}, \mathcal{H} and \mathcal{P} either simplifies or factors. We shall return to this in future work.

¹³ On the other hand, the same degeneracy implies that exactly the same procedure (and equations) can be used for spaces of other metric signatures; that is, the procedure to give the discriminants for other signature metrics is therefore identical to the one given here, see [11].

2.2.4 Type II/D in higher dimensions

In higher dimensions we will obtain similar syzygies for type **II/D** tensors. In n dimensions, the Ricci and Weyl type **II/D** conditions are the corresponding syzygies ($m = n(n-1)/2$):

$$\text{Ricci:} \quad {}^n D_n = 0, \quad (21)$$

$$\text{Weyl:} \quad {}^m D_m = {}^m D_{m-1} = \dots = {}^m D_{m-n+2} = 0. \quad (22)$$

Note that the Ricci syzygy is of order $n(n-1)$, while the highest Weyl syzygy is of order $n(n^2-1)(n-2)/4$.

2.3 Examples

The \mathcal{CHP} conditions are non-trivial: Let us consider a simple metric which shows the \mathcal{CHP} conditions are non-trivial conditions. Consider the solvmanifold:

$$ds^2 = -dt^2 + e^{2p_1 t} dx^2 + e^{2p_2 t} dy^2 + e^{2p_3 t} dz^2 + e^{2p_4 t} dw^2 \quad (23)$$

The computation for this metric is a bit lengthy (even for MAPLE); however, by choosing randomly some values of the parameters¹⁴, for example, $p_1 = 1$, $p_2 = 2$, $p_3 = 5$, and $p_4 = -7$, MAPLE quickly calculates the values of the discriminants for the Weyl operator, obtaining:

$${}^{10}D_{10} > 0, \quad {}^{10}D_9 > 0, \quad {}^{10}D_8 > 0, \quad \dots, \quad {}^{10}D_2 > 0, \quad (24)$$

showing that this metric is not type **II** or **D** (and that the \mathcal{CHP} invariants are not trivial).

2.3.1 Some simple examples

Einstein spaces In an Einstein space all of $\text{Tr}(\mathbf{R}^k) \propto \Lambda^k$, and hence all of the discriminants for the trace-free Ricci eigenvalue equation are zero, and the only non-trivial scalar invariant is the Ricci scalar $R \propto \Lambda$.

5D Schwarzschild spacetime: For the Weyl operator \mathbf{C} we get

$${}^{10}D_{10} = {}^{10}D_9 = \dots = {}^{10}D_4 = 0, \quad {}^{10}D_3 > 0, \quad {}^{10}D_2 > 0.$$

This implies that the Weyl operator has 3 distinct real eigenvalues which agrees with the results of [7]. In fact, this spacetime is of type **D**.

5D space with complex hyperbolic sections. Let us consider the example in [7] with complex hyperbolic spatial sections:

$$ds^2 = -dt^2 + a(t)^2 \left[e^{-2w} \left(dx + \frac{1}{2}(ydz - zdy) \right)^2 + e^{-w} (dy^2 + dz^2) + dw^2 \right]. \quad (25)$$

¹⁴For special values of the parameters this metric has some symmetries; however, this is not generally the case.

For the Weyl operator \mathbf{C} we get

$${}^{10}D_{10} = {}^{10}D_9 = \dots = {}^{10}D_4 = 0, \quad {}^{10}D_3 > 0, \quad {}^{10}D_2 > 0.$$

This again implies that the Weyl operator has 3 distinct real eigenvalues, like the 5D Schwarzschild spacetime. However, unlike the Schwarzschild case, we note that the some of the invariants a_i , defined in eq.(9), are zero:

$$a_{10} = a_9 = a_8 = \dots = a_4 = 0.$$

As explained, this is a signal that there is a zero-eigenvalue of multiplicity 7! Hence, since there are 3 distinct eigenvalues of which one of must be zero with multiplicity 7, the eigenvalue structure is $\{(1111111)(11)1\}$. In particular, since the Weyl operator is trace-free, we explicitly get eigenvalues:

$$0 \text{ } [\times 7], \quad \lambda \text{ } [\times 2], \quad -2\lambda,$$

which agrees with the results of [7].

However, this spacetime is not of type **II/D**; indeed, it is \mathcal{I} -non-degenerate. This can be seen by computing the operator $T^\alpha_\beta = C^{\alpha\mu\nu\rho}C_{\beta\mu\nu\rho}$ which is of ‘Segre’ type $\{1, (1111)\}$. However, due to the fundamental degeneracy in the eigenvalue type, we need to compute differential invariants to delineate this case completely.

2.3.2 The rotating black ring.

The 5D rotating black ring [12] is generally of type **I_i**, but can also be of type **II** or **D** at different locations and for particular values of the parameters λ, μ . Assuming that the form of the metric is given by eqn. (9) in [12] (in terms of the parameters λ, μ , where R has been set to unity), we consider the coordinate ranges $-1 \leq x \leq 1$ and $1 \leq y < \infty$ (and hence $0 \leq \mu \leq 1$ and $0 \leq \lambda \leq 1$), corresponding to the regions B, A_2, A_3 in [12] in order to retain the correct (Lorentzian) signature. We consider the algebraic type of the 5D Weyl tensor. Calculating the polynomial invariants $\text{Tr}(\mathbf{C}^k)$ and evaluating at the ‘target’ point $x = 0$ and $y = 2$ in the region under consideration,¹⁵ all of the R_i and hence all of the resulting discriminants are functions of the parameters λ, μ only. Then, at the ‘target’ point, in general the metric is of type **I_i**, the case $\lambda = 1$ corresponds to the Myers-Perry metric (type **D**), $\mu = 1/2$ corresponds to the black hole horizon ($y = 1/\mu$, type **II**), $\mu = 0$ corresponds to the static subcase, and $y = 1/\lambda$ corresponds to a curvature singularity.

Let us first consider the trace-free part operator $T^\alpha_\beta = C^{\alpha\mu\nu\rho}C_{\beta\mu\nu\rho}$, which gives us the discriminant:

$${}^5_T D_5 = \frac{\lambda^{12}(\lambda - \mu)^{12}(2\mu - 1)^2(1 - \lambda)^4(1 + \lambda)^4}{(1 - 2\lambda)^{113}} F(\mu, \lambda) \quad (26)$$

where $F(\mu, \lambda)$ is a polynomial which is generically not zero. On the horizon $\mu = 1/2$, we see that ${}^5_T D_5 = 0$, and computing ${}^5_T D_4$ we get ${}^5_T D_4 > 0$ except for special values of λ . This is a signal that the metric is of type **II** on the horizon.

¹⁵ Note that as $x \rightarrow 1$ and $y \rightarrow 1$ we obtain flat space in this region, but this case will not be included here.

Indeed, at the horizon, $\mu = 1/2$, the computation simplifies and we can compute the \mathcal{CHP} invariants:

$$\mathcal{C} = \mathcal{H} = \mathcal{P} = 0,$$

while:

$$\begin{aligned} {}^{10}_W D_7 \propto & \frac{\lambda^{12}(2\lambda^3 - \lambda^2 - 8\lambda - 16)(\lambda^4 - 2\lambda^2 - 2\lambda + 1)}{(2\lambda - 1)^{83}} \\ & \times (\lambda - 1)^2(\lambda + 2)^2(\lambda + 1)^2(4\lambda^2 - \lambda - 6)^2(2\lambda^2 - \lambda - 4)^2 \\ & \times (4\lambda^3 - 4\lambda^2 - 3\lambda + 6)^2(2\lambda^4 - 12\lambda^3 + 9\lambda^2 + 16\lambda - 24)^2 \\ & \times (4\lambda^4 - 14\lambda^2 + \lambda + 12)^2(\lambda^3 - 2\lambda^2 + \lambda + 6)^2(2\lambda + 1)^6, \end{aligned} \quad (27)$$

where a (positive) numerical factor has been ignored. Since, the \mathcal{CHP} syzygies are satisfied, this gives further evidence that the metric is of type **II** on the horizon. Note that we actually get further constraints from the secondary discriminants, as can be seen from the table in Appendix B.4. Indeed, by calculating the secondary discriminant ${}^{10}_W F_3$ on the horizon, we get:

$${}^{10}_W F_3 \propto \frac{\lambda^2(\lambda^4 - 2\lambda^2 - 2\lambda + 1)(2\lambda^2 - \lambda - 4)^2}{(2\lambda - 1)^{12}}, \quad (28)$$

which we see is non-zero as long as ${}^{10}_W D_7 \neq 0$. Consequently, as long as ${}^{10}_W D_7 \neq 0$, then the eigenvalue type is $\{(11)(11)(11)1..1\}$. This is consistent with type **II**.

Another interesting special case is $\lambda = 1$ (Myers-Perry), for which both ${}^5_T D_5 = {}^5_T D_4 = 0$, and:

$${}^5_T D_3 = 67108864\mu^2(\mu - 1)^2(\mu - 5)^2(5\mu - 1)^2(\mu + 1)^2, \quad {}^5_T D_2 > 0.$$

We can also here compute the \mathcal{CHP} invariants, which are all zero.

Note: We note that ${}^5_T D_5 = 0$ is a 40th order syzygy in the Weyl tensor. Therefore, a useful strategy in practical computations (for example, determining the algebraic type of a 5D Weyl tensor), as illustrated by this example, might be to test for necessity using an operator like \mathbb{T} , which is relatively simple. If the syzygy is not satisfied we have a definitive result. It is possible that the syzygy can only be satisfied for certain coordinate values (or parameter values), whence the \mathcal{CHP} syzygies can be tested in these simpler particular cases.

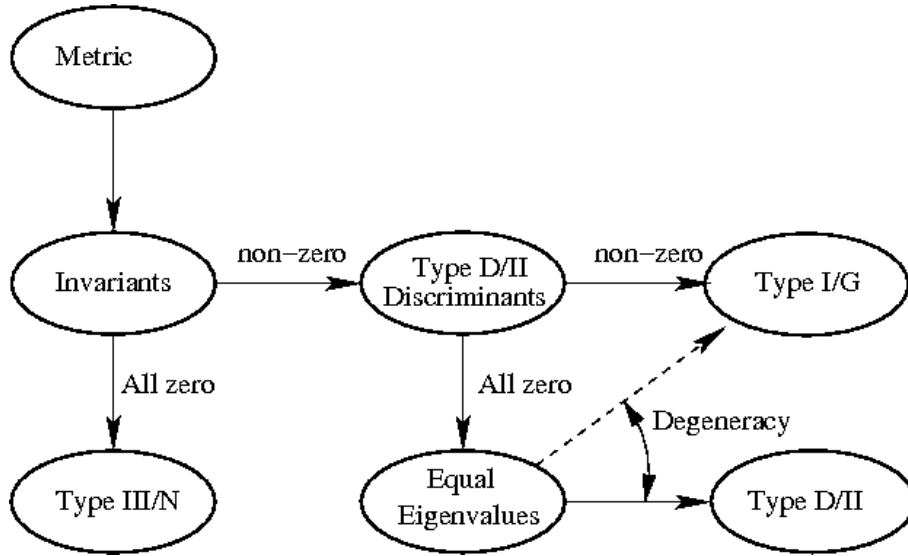


Figure 1: A flow diagram indicating how we can attempt to determine the algebraic type for tensors. The degeneracy indicated is due to the fact that several types may have the same eigenvalue type (sometimes this can be resolved by considering other invariants, but not always).

3 Conclusions

3.1 Discussion

For a curvature tensor of a particular algebraic type, the associated operator will have a restricted eigenvector structure. For a given curvature operator in arbitrary dimensions, we can thus consider the eigenvalues of this operator to obtain necessary conditions in order for the tensor to be of a particular algebraic type. In principle, this analysis can be used to study all of the various subclasses of particular algebraic types in more detail. In particular, requiring the algebraic type to be **II** or **D** will impose useful restrictions. In this paper we have used an analysis of the discriminants of the associated characteristic equation to determine the conditions on a tensor for a given algebraic type. Since the characteristic equation has coefficients which can be expressed in terms of the scalar polynomial curvature invariants of the operator, we can give conditions, or syzygies, on the eigenvalue structure expressed manifestly in terms of these polynomial scalar curvature invariants. Indeed, we have described an algorithm which enables us to completely determine the eigenvalue structure of the curvature, up to degeneracies, in terms of a set of *discriminants* ${}^n D_i$, ${}^n E_i$, etc.. The resulting syzygies (discriminants) can then be written as special scalar polynomial invariants.

In particular, we have used the technique to study the necessary conditions in arbitrary dimensions for the Weyl and Ricci curvature operators (and hence the higher dimensional Weyl and Ricci tensors) to be of algebraic type **II/D**, and created syzygies which are necessary for the special algebraic type to be fulfilled. We are consequently able to determine the necessary conditions in terms

of simple scalar polynomial curvature invariant for the higher dimensional Weyl and Ricci tensors to be of type **II** or **D**. We have explicitly determined the scalar polynomial curvature invariants for a Weyl or Ricci tensor to be of type **II** (or **D**) in 5D. This will be of considerable utility in classifying higher dimensional solutions obtained in supergravity or superstring theory [3] or higher dimensional black hole solutions [4].

3.1.1 Summary of results.

A number of specific results have been obtained in this work. The necessary condition for the trace-free Ricci tensor, S , to be of algebraic type **II** (or **D**) in 5D is that the discriminant ${}^5_S D_5$ is zero, so that the related scalar polynomial curvature invariant $\mathcal{D} \equiv {}^5_S D_5 = 0$. The necessary condition for the Weyl tensor to be of type **II** (or **D**) in 5D is that the scalar polynomial curvature invariants $\mathcal{C} = \mathcal{H} = \mathcal{P} = 0$. In principle, we can repeat a similar analysis for other algebraic types (and making more use of the secondary discriminants).

A number of simple examples were presented, including Einstein spaces, the 5D Schwarzschild spacetime, and 5D space with complex hyperbolic sections. In addition, a simple solvmanifold was considered to show that the \mathcal{CHP} conditions are non-trivial.

We also presented a detailed analysis of the important example of a 5D rotating black ring [12] which is generally of type **I₁**, but can also be of type **II** or **D** for particular values of the parameters. This example serves to illustrate the calculational method and the power of the approach. In particular, we showed that the rotating black ring is of type **II** (or type **D**) on the black hole horizon ($y = 1/\mu$), by showing that $\mathcal{C} = \mathcal{H} = \mathcal{P} = 0$ on the horizon (and studying some of the secondary discriminants). The example also illustrates the utility in practical computations of employing rather more simple discriminants like ${}^5_T D_5 = 0$ (which is a 40th order syzygy in the Weyl tensor).

3.2 Classification using scalar invariants

In Lorentzian spacetimes, identical metrics are often given in different coordinate systems, which disguises their true equivalence. Perhaps the easiest way of distinguishing metrics is through their *scalar polynomial curvature invariants*, due to the fact that inequivalent invariants implies inequivalent metrics. In [9] the notion of an \mathcal{I} -non-degenerate spacetime metric in the class of 4D Lorentzian manifolds, which implies that the spacetime metric is locally determined by its scalar polynomial curvature invariants, was introduced. By determining an appropriate set of projection operators from the Riemann tensor and its covariant derivatives, it was proven that a 4D Lorentzian spacetime metric is either \mathcal{I} -non-degenerate or *degenerate Kundt* [9]. Therefore, a metric that is not characterized by its curvature invariants must be of degenerate Kundt form. These results were generalized to higher dimensions in [15].

3.2.1 Differential invariants

The \mathcal{I} -non-degenerate theorem contains not only zeroth order invariants but also differential scalar polynomial curvature invariants constructed from the Riemann tensor and its covariant derivatives. For example, if the spacetime is

of Weyl type **N**, then the differential invariants \mathcal{I}_1 and \mathcal{I}_2 vanish if the spacetime is degenerate Kundt [6] (the definitions of the invariants \mathcal{I}_1 and \mathcal{I}_2 are given therein). Similar results follow for Weyl type **III** spacetimes (in terms of invariants $\tilde{\mathcal{I}}_1$ and $\tilde{\mathcal{I}}_2$) and in the conformally flat (but non-vacuum) case (in terms of similar invariants \mathcal{I}_1 and \mathcal{I}_2 constructed from the Ricci tensor [6]).

These conditions are *necessary* conditions in order for a spacetime not to be \mathcal{I} -non-degenerate [9]. In the case that $27J^2 = I^3 \neq 0$ (Weyl types **II** or **D**), in [9] two higher order invariants S_1 and S_2 were given as *sufficient* conditions for \mathcal{I} -non-degeneracy (if $27J^2 = I^3$, but $S_1 \neq 0$ or $S_2 \neq 0$, then the spacetime is \mathcal{I} -non-degenerate).¹⁶

This analysis in 4D can be repeated using discriminants. Let us focus on the Ricci tensor for illustrative purposes. The necessary condition for the Ricci tensor to be of type **II** or **D** is given by eqn. (4) which, as noted earlier, follows from a discriminant analysis. Now, if we consider the covariant derivatives of the Ricci tensor, $R_{ab;cd}\dots$, then for the spacetime to be \mathcal{I} -non-degenerate each covariant derivative term must also be of type **II** or **D**. Hence we could study the eigenvalue structure of the operators associated with the $R_{ab;cd}\dots$ and apply the type **II/D** necessary conditions in turn. For example, considering the trace-free parts of the tensors $T_{ab} = R_{ac;d}R_b{}^{c;d}, R_{;ab}, \square R_{ab}, \dots$, we obtain necessary conditions of the form of eqn. (4) but with the $s_i \equiv \text{Tr}(\mathbf{T}^i), i = 2, 3, 4$.¹⁷

This can be repeated for the Weyl tensor and in higher dimensions [15].

Example. The class of vacuum type **D** spacetimes which are \mathcal{I} -non-degenerate, are invariantly classified by their scalar polynomial curvature invariants [16]. For example, for the Kinnersley class I type **D** vacuum spacetime [17] (the other cases work in a similar way), there are 4 algebraically independent (complex) Cartan invariants, which can be written in terms of 4 independent (complex) scalar polynomial invariants (e.g., I , and invariants such as $T^\mu{}_\mu$, where

$$T^\mu{}_\nu \equiv C^{\alpha\beta\gamma\delta;\mu} C_{\alpha\beta\gamma\delta;\nu}$$

which include first and second covariant derivatives). The Schwarzschild vacuum type **D** spacetime belongs to the Kinnersley class I and, as discussed in [9], all of the algebraically independent Cartan scalars are related to the two functionally independent polynomial scalar curvature invariants $C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}$ and $T^\mu{}_\mu$ (which are equal to $48M^2r^{-6}$ and $720(r-2M)M^2r^{-9}$, respectively, as functions of the two parameters r and M in canonical coordinates) [18]. The Kerr solution belongs to Kinnersley class IIA; this spacetime has been invariantly characterized intrinsically [19].

Kerr metric. For illustration let us consider the example of the Kerr metric. The Kerr metric is of Petrov type **D** and we are thus interested in whether the covariant derivative, ∇C , is of type **D** or not. As noted above, the methods discussed earlier can also be used for differential invariants. The covariant derivative of the Weyl tensor, $C_{\alpha\beta\gamma\delta;\mu}$, has no natural operator associated with

¹⁶ If the spacetime is \mathcal{I} -non-degenerate, then essentially we can construct positive boost weight terms in the derivatives of the curvature and determine an appropriate set of differential scalar curvature invariants.

¹⁷ In practice it may be advantageous to work with operators involving second covariant derivatives.

it. However, we can, for example, consider the second order operator T^μ_ν defined above. If T^μ_ν is not of type **D/II**, then ∇C cannot be of type **D/II** either.

For the Kerr metric, we obtain the syzygy:

$${}_TD_4 = \frac{m^{24} a^4 G_-^2 G_+^2 (r^2 + a^2 - 2mr)^2 (r^2 + a^2 \cos^2 \theta - 2mr)^2 \sin^4 \theta}{(r^2 + a^2 \cos^2 \theta)^{92}} f_1^2 f_2,$$

where

$$G_\pm = r^4 \pm 4ar^3 \cos \theta - 6a^2 r^2 \cos^2 \theta \mp 4a^3 r \cos^3 \theta + a^4 \cos^4 \theta, \quad (29)$$

and $f_1 = f_1(a, m, r, \cos \theta)$ and $f_2 = f_2(a, m, r, \cos \theta)$ are some complicated polynomials. We note that away from the horizon, the ergosphere, and some other special points, this syzygy is non-zero and hence ∇C is not of type **D/II** (generically), outside the horizon. The Kerr metric is therefore \mathcal{I} -non-degenerate by the results of [9].

3.3 Physical applications

Recently, there has been considerable interest in black holes in more than four dimensions [4]. While the study of black holes in higher dimensions was perhaps originally motivated by supergravity and string theory, now the physical properties of such black holes are of interest in their own right. Indeed, studies have shown that even at the classical level gravity in higher dimensions exhibits much richer dynamics than in 4D, and one of the most remarkable features of higher dimensions is the non-uniqueness of black holes [4].

There now exist a number of different higher dimensional black hole solutions [4], including the rotating black rings [12], that are the subject of ongoing research in classical relativity and string theory. Some of these new spacetimes have been classified algebraically [5, 12]. However, in order to make further progress it is absolutely crucial to be able to develop new techniques for solving the vacuum field equations in higher dimensions and to be able to comprehensively classify such solutions, and the algebraic techniques recently introduced [5, 7] will be of fundamental importance in this development. However, the algebraic techniques used to date now are rather difficult to apply, and the development of simpler criteria, including the use of necessary conditions in terms of scalar curvature invariants introduced here, will hopefully prove to be of great utility.

Therefore, the analysis presented in this paper will be of considerable importance for analysing higher dimensional black hole solutions [4] (and solutions in supergravity or superstring theory [3]). Indeed, the detailed analysis of the 5D rotating black ring [12] serves to illustrate the power of the approach.

In future work we hope to extend the analysis presented in this paper and further generalize it to the study of differential operators. In addition, we intend to discuss a number of other applications, including the algebraic classification of some other known higher dimensional black hole solutions.

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A The Weyl Bivector operator

Given a vector basis \mathbf{k}^μ we can define a set of (simple) bivectors

$$\mathbf{F}^A \equiv \mathbf{F}^{\mu\nu} = \mathbf{F}^{[\mu\nu]} = \mathbf{k}^\mu \wedge \mathbf{k}^\nu,$$

spanning the space of antisymmetric tensors of rank 2. Consider a $D = (2+n)$ -dimensional Lorentzian space with the following null-frame $\{\ell, \mathbf{n}, \mathbf{m}^i\}$ so that the metric is

$$ds^2 = 2\ell\mathbf{n} + \delta_{ij}\mathbf{m}^i\mathbf{m}^j.$$

Let us consider the following bivector basis:

$$\ell \wedge \mathbf{m}^i, \quad \ell \wedge \mathbf{n}, \quad \mathbf{m}^i \wedge \mathbf{m}^j, \quad \mathbf{n} \wedge \mathbf{m}^j,$$

or for short: $[0i]$, $[01]$, $[ij]$, $[1i]$. The Lorentz metric also induces a metric, η_{MN} , in bivector space. If $m = n(n-1)/2$, then

$$(\eta_{MN}) = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 1_n \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1_m & 0 \\ 1_n & 0 & 0 & 0 \end{bmatrix},$$

where 1_n , and 1_m are the unit matrices of size $n \times n$ and $m \times m$, respectively, and we have assumed the bivector basis is in the order given above. This metric can then be used to raise and lower bivector indices.

Let $V \equiv \wedge^2 T_p^* M$ be the vector space of bivectors at a point p . Then consider an operator $\mathbf{C} = (C_N^M) : V \mapsto V$. We will assume that it is symmetric in the sense that $C_{MN} = C_{NM}$. With these assumptions, the operator \mathbf{C} can be written in the following $(n+1+m+n)$ -block form [7]:

$$\mathbf{C} = \begin{bmatrix} M & \hat{K} & \hat{L} & \hat{H} \\ \check{K}^t & -\Phi & -A^t & -\hat{K}^t \\ \check{L}^t & A & \bar{H} & \hat{L}^t \\ \check{H} & -\check{K} & \check{L} & M^t \end{bmatrix} \quad (30)$$

Here, the block matrices H (barred, checked and hatted) are all symmetric. Checked (hatted) matrices correspond to negative (positive) boost weight components.

The eigenbivector problem can now be formulated as follows. A bivector F_A is an eigenbivector of \mathbf{C} if and only if

$$C_N^M F_M = \lambda F_N, \quad \lambda \in \mathbb{C}.$$

Such eigenbivectors can now be determined using standard results from linear algebra. The Lorentz transformations (boosts, spins and null rotations) in $(2+n)$ -dimensions are explicitly written down in [7].

A.1 Weyl operator

In particular, for the Weyl tensor we can make the following identifications (indices B, C, \dots should be understood as indices over $[ij]$):

$$\hat{H}^i_j = C_{0i0j}, \quad \check{H}^i_j = C_{1i1j}, \quad (31)$$

$$\hat{L}^i_B = C_{0ijk}, \quad \check{L}^i_B = C_{1ijk}, \quad (32)$$

$$\hat{K}^i = C_{010i}, \quad \check{K}^i = -C_{011i}, \quad (33)$$

$$M^i_j = C_{1i0j}, \quad \Phi = C_{0101}, \quad (34)$$

$$A^B = C_{01ij}, \quad \bar{H}^B_C = C_{ijkl}. \quad (35)$$

The Weyl tensor is also traceless and obeys the Bianchi identity:

$$C^\mu_{\alpha\mu\beta} = 0, \quad C_{\alpha(\beta\mu\nu)} = 0.$$

These conditions translate into conditions on our block matrices. We can consider each boost weight in turn, and use this to express these matrices into irreducible representations of the spins [7].

A.1.1 Boost-weight 0 components

Here we have

$$C_{0101} = C_{0i1}^i, \quad C_{0i1j} = -\frac{1}{2}C_{ikj}^k + \frac{1}{2}C_{01ij}, \quad C_{i(jkl)} = 0. \quad (36)$$

Starting with the latter, this means that the matrix \bar{H}^B_C fulfills the reduced Bianchi identities. It is also symmetric which means that it has the same symmetries as an n -dimensional Riemann tensor. Hence, we can split this into irreducible parts over $SO(n)$ using the ‘‘Weyl tensor’’, ‘‘trace-free Ricci’’ and ‘‘Ricci scalar’’ as follows ($n > 2$):

$$\bar{H}_{BC} = \bar{C}_{ijkl} + \frac{2}{n-2}(\delta_{i[k}\bar{R}_{l]j} - \delta_{j[k}\bar{R}_{l]i}) - \frac{2}{(n-1)(n-2)}\bar{R}\delta_{i[k}\delta_{l]j}, \quad (37)$$

$$\bar{R}_{ij} = \bar{S}_{ij} + \frac{1}{n}\bar{R}\delta_{ij}. \quad (38)$$

The remaining Bianchi identities now imply:

$$M_{ij} = -\frac{1}{2n}\bar{R}\delta_{ij} - \frac{1}{2}\bar{S}_{ij} - \frac{1}{2}A_{ij} \quad (39)$$

$$\Phi = -\frac{1}{2}\bar{R}. \quad (40)$$

This means that the boost weight 0 components can be specified using the irreducible compositions above ($\bar{R}, \bar{S}_{ij}, A_{ij}, \bar{C}_{ijkl}$). We note that in lower dimensions we have the special cases for the n -dimensional Riemann tensor: (i) Dim 4 ($n = 2$): $\bar{S}_{ij} = \bar{C}_{ijkl} = 0$, (ii) Dim 5 ($n = 3$): $\bar{C}_{ijkl} = 0$, (iii) Dim 6 ($n = 4$): $\bar{C}_{ijkl} = \bar{C}_{ijkl}^+ + \bar{C}_{ijkl}^-$, where \bar{C}^+ and \bar{C}^- are the self-dual, and the anti-self-dual parts of the Weyl tensor, respectively. The same can be done with the antisymmetric tensor $A_{ij} = A_{ij}^+ + A_{ij}^-$.

A spin $G \in SO(n)$ acts as follows on the various matrices:

$$(M, \Phi, A, \bar{H}) \mapsto (GMG^{-1}, \Phi, \bar{G}A, \bar{G}\bar{H}\bar{G}^{-1}). \quad (41)$$

If $C_{\mu\nu\alpha\beta}$ is the Weyl tensor, the type **D** case is therefore completely characterised in terms of a n -dimensional Ricci tensor, a Weyl tensor, and an antisymmetric tensor A_{ij} . Therefore, the spins are first used to diagonalise the “Ricci tensor” \bar{R}_{ij} . This matrix can then be described in terms of the Segre-like notation corresponding to its eigenvalues. There is a degeneracy in the eigenvalues which occurs when two, or more, eigenvalues are equal. Using a Segre-like notation, we therefore get the types for \bar{R}_{ij} :

$$\{1111..\}, \{(11)11..\}, \{(11)(11)...\}, \text{etc.}, \{0111..\}, \{0(11)1..\}, \{00(11)...\}, \text{etc.}, \quad (42)$$

where a zero indicates a zero-eigenvalue. Regarding the antisymmetric matrix A_{ij} , this must be of even rank and can be put into canonical block-diagonal form, and we can characterise an antisymmetric matrix using the rank. The antisymmetric matrix A may also possess further degeneracies. Finally, characterisation of the “Weyl tensor” \bar{C}_{ijkl} reduces to characterising the Weyl tensor of the corresponding fictitious n -dimensional Riemannian manifold.

A.2 The algebraic classification

Let us consider the classification in [5], and investigate the different algebraic types in turn. In general, there will be algebraically special cases of type **G**. The type **I**, **III** and **N**’s were delineated in [7]. It is of interest to explicitly review the type **II/D**’s here.

A.2.1 Type **II/D**

The tensor $C_{\mu\nu\alpha\beta}$ is of type **II/D** if and only if there exists a null frame such that the operator **C** takes the form:

$$\mathbf{C} = \begin{bmatrix} M & 0 & 0 & 0 \\ \check{K}^t & -\Phi & -A^t & 0 \\ \check{L}^t & A & \bar{H} & 0 \\ \check{H} & -\check{K} & \check{L} & M^t \end{bmatrix} \quad (43)$$

For type **D** there exists a null frame such that, in addition, $\check{K}^t = 0, \check{L}^t = 0, \check{H} = 0, \check{K} = 0, \check{L} = 0$.¹⁸

Then there will be algebraic subcases according to whether some of the irreducible components of boost weight 0 are zero or not. A complete characterisation of all such subcases is very involved in its full generality. However, a rough classification in terms of the vanishing the irreducible components under spins can be made: (a) Type **II/D**(a): $A = 0$, (b) Type **II/D**(b): $\bar{R}_{ij} = 0$, (c) Type **II/D**(c): $\bar{C}_{ijkl} = 0$. Note that we can also have a combination of these; for example, type **II**(ac), which means that $A = 0$ and $\bar{C} = 0$ (i.e., further algebraically special subcases can arise).

¹⁸ For type **D** tensors, which are invariant under boosts, all Lorentz transformations has been utilised *except* for the spins.

A.3 Type D in 4D ($n = 2$)

In 4D, the Weyl operator can always be put into type I form by using a null rotation (hence, $\hat{H} = 0$). Furthermore, the irreducible representations under the spins are: $\hat{v}^i, \bar{R}, A, \check{v}^i, \check{H}$. Utilizing the unused freedom of one spin, one boost and two null-rotations, in each of the algebraically special cases we can use these to simplify the Weyl tensor even further.

Let us only consider type **D** for illustration. For $n = 2$, the Weyl tensor reduces to specifying two scalars, namely \bar{R} and A_{34} . We now get

$$M = \begin{bmatrix} -\frac{1}{4}\bar{R} & -\frac{1}{2}A_{34} \\ \frac{1}{2}A_{34} & -\frac{1}{4}\bar{R} \end{bmatrix}, \quad \begin{bmatrix} -\Phi & -A^t \\ A & \bar{H} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\bar{R} & -A_{34} \\ A_{34} & \frac{1}{2}\bar{R} \end{bmatrix}; \quad (44)$$

consequently, the Weyl operator C has eigenvalues:

$$\lambda_{1,2} = \lambda_{3,4} = -\frac{1}{4}(\bar{R} \pm 2iA_{34}), \quad \lambda_{5,6} = \frac{1}{2}(\bar{R} \pm 2iA_{34}). \quad (45)$$

We note that this is in agreement with the standard type **D** analysis in 4D (see [8]). The type **D** case is boost invariant, and also invariant under spins, consequently the isotropy is 2-dimensional.

The two subcases $A_{34} = 0$ and $\bar{R} = 0$ (type **D(a)** and **D(b)**, respectively) are in 4D referred to the purely “electric” and “magnetic” cases, respectively. In 4D, there is a duality relation, \star , which interchanges these two cases; i.e., $C \mapsto \star C$ interchanges the electric and magnetic parts.

A.4 Type II/D in 5D ($n = 3$)

The 5D case is considerably more difficult than the 4D case. The complexity drastically increases and hence the number of special cases also increases. However, the 5D case is still manageable and some simplifications occur (compared to the general case). Most notably, $\bar{C}_{ijkl} = 0$, and \bar{T}_{jk}^i can be written, using a matrix \check{n}_{ij} , as follows (similarly for \hat{T}_{jk}^i):

$$\bar{T}_{jk}^i = \varepsilon_{jkl} \check{n}^{li}, \quad (46)$$

where the conditions on \bar{T}_{jk}^i imply that \check{n}^{ij} is symmetric and trace-free. Therefore, we can use the spins to diagonalise \check{n}^{ij} . Thus the general case is $\{111\}$ (all eigenvalues different), with the special cases $\{(11)1\}$, $\{110\}$ and $\{000\}$. Furthermore, in the general case, the vector \check{v}^i needs not be aligned with the eigenvectors of \check{n}^{ij} . There would consequently be special cases where \check{v}^i is an eigenvector of \check{n}^{ij} . The components in 5D are displayed in Table 1. For type **D** we have that $\check{n}^{ij} = 0$, and hence $\bar{T}_{jk}^i = 0$.

A.4.1 Type D

For a type **D** Weyl tensor only the following components can be non-zero:

$$\bar{R}, \quad \bar{S}_j^i, \quad A_{ij},$$

boost weight	Ind. Components	Weyl components
+2	\hat{H}_{ij}	$C_{0i0j} = \hat{H}_{ij}$
+1	\hat{v}_i, \hat{n}_{ij}	$C_{0ijk} = \delta_{ij}\hat{v}_k - \delta_{ik}\hat{v}_j + \varepsilon_{jkl}\hat{n}^{li}, \quad C_{010i} = 2\hat{v}_i$
0	$\bar{R}, \bar{S}_{ij}, A_{ij}$	$\begin{cases} C_{1i0j} = -\frac{1}{2}\bar{R}_{ij} - \frac{1}{2}A_{ij}, & C_{01ij} = A_{ij}, \\ C_{0101} = -\frac{1}{2}\bar{R}, & C_{ijkl} = \bar{R}_{ijkl} \end{cases}$
-1	$\check{v}_i, \check{n}_{ij}$	$C_{1ijk} = \delta_{ij}\check{v}_k - \delta_{ik}\check{v}_j + \varepsilon_{jkl}\check{n}^{li}, \quad C_{011i} = -2\check{v}_i$
-2	\check{H}_{ij}	$C_{0i0j} = \check{H}_{ij}$

Table 1: Dimension $D = 5$: Here $\bar{R}_{ikj}^k = \bar{R}_{ij} = \frac{1}{3}\bar{R}\delta_{ij} + \bar{S}_{ij}$. Also see [16]

where $i, j = 3, 4, 5$. Let us use the spins to diagonalise $(S_j^i) = \text{diag}(S_{33}, S_{44}, S_{55})$. Without any further assumptions, the Weyl blocks take the form:

$$\begin{aligned}
 M &= \begin{bmatrix} -\frac{1}{6}\bar{R} - \frac{1}{2}S_{33} & -\frac{1}{2}A_{34} & \frac{1}{2}A_{53} \\ \frac{1}{2}A_{34} & -\frac{1}{6}\bar{R} - \frac{1}{2}S_{44} & -\frac{1}{2}A_{45} \\ -\frac{1}{2}A_{53} & \frac{1}{2}A_{45} & -\frac{1}{6}\bar{R} - \frac{1}{2}S_{55} \end{bmatrix}, \\
 \begin{bmatrix} -\Phi & -A^t \\ A & \bar{H} \end{bmatrix} &= \begin{bmatrix} \frac{1}{2}\bar{R} & -A_{45} & -A_{53} & -A_{34} \\ A_{45} & \frac{1}{6}\bar{R} - S_{33} & 0 & 0 \\ A_{53} & 0 & \frac{1}{6}\bar{R} - S_{44} & 0 \\ A_{34} & 0 & 0 & \frac{1}{6}\bar{R} - S_{55} \end{bmatrix} \quad (47)
 \end{aligned}$$

The general type **D** tensor thus has this canonical form.

There are two special cases where we can use the extra symmetry to get the simplified canonical form: (i) $S_{33} = S_{44} = -2S_{55}$: $A_{35} = 0$. (ii) $S_{33} = S_{44} = S_{55} = 0$: $A_{35} = A_{45} = 0$. We note that case (ii) will, without further assumptions, be invariant under spatial rotations in the [34]-plane (in addition to the boost). Assuming, in addition, that $A_{45} = 0$, then case (i) is also invariant under a rotation in the [34]-plane. Assuming that A_{ij} vanishes completely, we note that case (ii) enjoys the full invariance under the spins (i.e., $SO(3)$).

B Some Discriminants

For convenience, let us consider a trace-free operator S so that $S_I = \text{Tr}(S) = 0$. We recall that $S_i \equiv \text{Tr}(S^i)$. We also will give the table that gives the eigenvalue type using these discriminants. Here, $1_{\mathbb{C}}$ means a pair of complex conjugate eigenvalues.

To translate into Ricci/Weyl type we need to consider degeneracies. For example, the Eigenvalue type $\{(11)11\}$, corresponds to the three Ricci types $\{(1,1)11\}$, $\{1,1(11)\}$, $\{211\}$ because all of these have the same eigenvalue type.

B.1 Dimension 3 Operator

For a 3-dimensional trace-free operator ($S_I = 0$), the syzygies are given by:

$$\begin{aligned}
 {}^3D_2 &= 3S_2 \\
 {}^3D_3 &= \frac{1}{2}S_2^3 - 3S_3^2 \quad (48)
 \end{aligned}$$

3D_3	3D_2	Eigenvalue type
+	+	$\{111\}$
-		$\{1_{\mathbb{C}}1\}$
0	+	$\{(11)1\}$
0	0	$\{(111)\}$

B.2 Dimension 4 Operator

For a 4-dimensional trace-free operator ($S_I = 0$), the syzygies are given by:

$$\begin{aligned}
{}^4D_2 &= 4S_2 \\
{}^4D_3 &= -S_2^3 + 4S_2S_4 - 4S_3^2 \\
{}^4D_4 &= \frac{1}{8}S_2^6 - \frac{5}{4}S_2^4S_4 - \frac{17}{18}S_3^2S_2^3 \\
&\quad + 4S_2^2S_4^2 + 2S_3^2S_2S_4 - \frac{1}{3}S_3^4 - 4S_4^3 \\
{}^4E_2 &= S_3^2 + 2S_2^3 - 4S_2S_4
\end{aligned} \tag{49}$$

4D_4	4D_3	4D_2	4E_2	Eigenvalue type
+	+	+		$\{1111\}$
+	$\pm/0$	$\mp/0$		$\{1_{\mathbb{C}}1_{\mathbb{C}}\}$
-	$\pm/0$	$\pm/0$		$\{1_{\mathbb{C}}11\}$
0	+	+		$\{(11)11\}$
0	\pm	\mp		$\{(11)1_{\mathbb{C}}\}$
0	0	+	+	$\{(11)(11)\}$
0	0	+	0	$\{(111)1\}$
0	0	-		$\{(1_{\mathbb{C}}1_{\mathbb{C}})\}$
0	0	0		$\{(1111)\}$

B.3 Dimension 5 Operator

For a 5-dimensional trace-free operator ($S_1 = 0$), the syzygies are given by:

$$\begin{aligned}
{}^5D_2 &= 5S_2 \\
{}^5D_3 &= -S_2^3 + 5S_2S_4 - 5S_3^2 \\
{}^5D_4 &= \frac{1}{8}S_2^6 - \frac{11}{8}S_2^4S_4 + \frac{7}{24}S_2^3S_3^2 \\
&\quad + 2S_2^2S_3S_5 + \frac{19}{4}S_2^2S_4^2 - \frac{61}{12}S_2S_3^2S_4 \\
&\quad - 5S_2S_5^2 - \frac{2}{3}S_3^4 - 5S_4^3 + 10S_3S_5S_4 \\
{}^5D_5 &= \frac{21}{2}S_2^2S_3^2S_5^2 - \frac{539}{120}S_2^3S_3^3S_5 - \frac{91}{72}S_2S_3^2S_4^3 - \frac{31}{96}S_2^3S_3^2S_4^2 \\
&\quad + \frac{41}{96}S_2^5S_3^2S_4 - \frac{5}{2}S_2S_5^2S_4^2 + \frac{11}{8}S_5^2S_2^3S_4 - \frac{59}{48}S_3^4S_4S_2^2 \\
&\quad + \frac{11}{48}S_2^6S_3S_5 + \frac{9}{4}S_3S_5S_2^2S_4^2 - \frac{31}{20}S_3S_5S_2^4S_4 + \frac{4}{45}S_3^5S_5 \\
&\quad - \frac{5}{2}S_3^2S_4S_5^2 - \frac{2}{27}S_3^6S_2 - \frac{35}{3}S_2S_3S_5^3 + \frac{1}{512}S_2^{10} \\
&\quad - \frac{1}{48}S_3^4S_4^2 - \frac{79}{400}S_2^5S_5^2 - \frac{79}{1152}S_2^7S_3^2 \\
&\quad + \frac{151}{192}S_2^4S_3^4 - \frac{7}{256}S_2^8S_4 + \frac{19}{128}S_2^6S_4^2 + \frac{5}{3}S_3S_5S_4^3 \\
&\quad - \frac{25}{64}S_2^4S_4^3 + \frac{1}{2}S_2^2S_4^4 - \frac{1}{4}S_4^5 + 5S_5^4 + \frac{43}{12}S_3^3S_4S_2S_5 \\
{}^5E_2 &= \frac{91}{24}S_2^4S_3^2 - \frac{15}{2}S_2^3S_3S_5 \\
&\quad + \frac{25}{4}S_2^2S_5^2 - \frac{205}{12}S_4S_2^2S_3^2 + \frac{25}{2}S_4S_2S_3S_5 - 25S_3^3S_5 \\
&\quad + \frac{25}{4}S_3^2S_4^2 - \frac{1}{8}S_2^7 + \frac{7}{8}S_2^5S_4 - \frac{5}{4}S_2^3S_4^2 + \frac{125}{6}S_3^4S_2 \\
{}^5F_2 &= \frac{1}{3}S_3^2 + \frac{1}{2}S_2^3 - S_2S_4
\end{aligned} \tag{50}$$

5D_5	5D_4	5D_3	5D_2	5E_2	5F_2	Eigenvalue type
+	+	+	+			{11111}
+	$\leq 0^*$	$\leq 0^*$	$\leq 0^*$			{1 _C 1 _C 1}
-						{1 _C 111}
0	+					{(11)111}
0	-					{(11)1 _C 1}
0	0	+		0		{(111)11}
0	0	-		0		{(111)1 _C }
0	0	+		$\neq 0$		{(11)(11)1}
0	0	-		$\neq 0$		{(1 _C 1 _C)1}
0	0	0	$\neq 0$		$\neq 0$	{(111)(11)}
0	0	0	$\neq 0$		0	{(1111)1}
0	0	0	0			{(11111)}

* One of these conditions is sufficient.

B.4 Dimension 10 Operator

For a 10-dimensional trace-free operator ($S_I = 0$), we only give a partial table indicating when the \mathcal{CHP} invariants are zero (we will also ignore whether the roots are real or complex).

$^{10}D_{10}$	$^{10}D_9$	$^{10}D_8$	$^{10}D_7$	$^{10}E_2$	$^{10}F_2$	$^{10}F_3$	Eigenvalue type
$\neq 0$							$\{111..1\}$
0	$\neq 0$						$\{(11)11...1\}$
0	0	$\neq 0$		$\neq 0$			$\{(11)(11)1...1\}$
0	0	$\neq 0$		0			$\{(111)1...1\}$
0	0	0	$\neq 0$			$\neq 0$	$\{(11)(11)(11)1..1\}$
0	0	0	$\neq 0$		$\neq 0$	0	$\{(111)(11)1..1\}$
0	0	0	$\neq 0$		0	0	$\{(1111)11..1\}$

B.4.1 The discriminant \mathcal{P}

The simplest of the three syzygies, the discriminant \mathcal{P} (which is of 56th order in terms of $S_k, k \leq 10$, and contains 13377 terms) is too lengthy to write down explicitly here,¹⁹ but it has the symbolic form:

$$\begin{aligned}
\mathcal{P} = & \frac{1}{56623104000} (S_6 S_8 - S_7^2) S_2^{21} + (\dots) S_2^{20} + \\
& \vdots \\
& + \dots + \dots + \\
& \vdots \\
& + (\dots) S_{10}^4 + (10S_4 S_2 - 10S_3^2 - S_2^3) S_{10}^5.
\end{aligned} \tag{51}$$

¹⁹ The explicit expression for \mathcal{P} is given in [20].

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